1a)
$$f(x)=(x+1)(x-2)^3$$
$$f'(x)=(x-2)^3+3(x+1)(x-2)^2$$
$$=(x-2)^2\left[x-2+3(x+1)\right]$$
$$=(x-2)^2(x-2+3x+3)$$
$$=(x-2)^2(4x+1)$$
$$f'(x)=0$$
$$x=2,-\frac{1}{4}$$

1b)
$$\frac{x^2}{y}+x=y-5$$
$$x^2+xy=y^2-5y$$
$$2x+y+x\frac{dy}{dx}=2y\frac{dy}{dx}-5\frac{dy}{dx}$$
$$2x+y=2y\frac{dy}{dx}-5\frac{dy}{dx}-x\frac{dy}{dx}$$
$$=\frac{dy}{dx}(2y-5-x)$$
$$\frac{dy}{dx}=\frac{2x+y}{2y-5-x}$$
$$=\frac{6-1}{-2-5-3}$$
$$=\frac{5}{-10}$$
$$=-\frac{1}{2}$$

2a)
$$A=\begin{pmatrix} t+4 & 3t \\ 3 & 5 \end{pmatrix}$$
$$A^{-1}=\frac{1}{\det A}\begin{pmatrix} 5 & -3t \\ -3 & t+4 \end{pmatrix}$$
$$\det A=5(t+4)-9t$$
$$=-4t+20$$
$$=4(5-t)$$
$$A^{-1}=\frac{1}{4(5-t)}\begin{pmatrix} 5 & -3t \\ -3 & t+4 \end{pmatrix}$$

2b)
$$\det A=4(5-t)=0$$
$$t=5$$

2c)
$$6=t+4=3t$$
$$t=2$$

3)
$$x^2e^y\frac{dy}{dx}=1$$
$$e^y dy=\frac{dx}{x^2}$$
$$\int e^y dy=\int\frac{dx}{x^2}$$
$$e^y=-\frac{1}{x}+C$$
$$e^0=1=-1+C$$
$$C=2$$
$$y=\ln\left|2-\frac{1}{x}\right|$$

4) $n=1$:
$$\text{LHS}=\frac{1}{1\times 2}=\frac{1}{2}$$
$$\text{RHS}=1-\frac{1}{1+1}=1-\frac{1}{2}=\frac{1}{2}=\text{LHS}$$

So it is true for $n=1$

Assume true for $n=k,\ k\in\mathbb{N}$ then:
$$\sum_{r=1}^{k}\frac{1}{r(r+1)}=1-\frac{1}{k+1}$$
Then for $n=k+1$:
$$\sum_{r=1}^{n}\frac{1}{r(r+1)}=\sum_{r=1}^{k+1}\frac{1}{r(r+1)}$$
$$=\sum_{r=1}^{k}\frac{1}{r(r+1)}+\sum_{r=k+1}^{k+1}\frac{1}{r(r+1)}$$
$$=1-\frac{1}{k+1}+\frac{1}{(k+1)(k+2)}$$
$$=1-\frac{k+2}{(k+1)(k+2)}+\frac{1}{(k+1)(k+2)}$$
$$=1+\frac{1-k-2}{(k+1)(k+2)}$$
$$=1+\frac{-k-1}{(k+1)(k+2)}$$
$$=1-\frac{k+1}{(k+1)(k+2)}$$
$$=1-\frac{1}{(k+2)}=1-\frac{1}{(k+1+1)}=1-\frac{1}{(n+1)}$$

Thus it is true for $n=k+1$ when true for $n=k$

Since it is also true for $n=1$ by induction it is true for all $n,\ n\in\mathbb{N}$

5)

$$I=\int_{\ln\frac{3}{2}}^{\ln 2}\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}dx$$

Let $f(x)=e^{x}-e^{-x}$ then $I=\int_{\ln\frac{3}{2}}^{\ln 2}\frac{f'(x)}{f(x)}dx$

$$\therefore I=\Big[\ln\left|f(x)\right|\Big]_{\ln\frac{3}{2}}^{\ln 2}=\Big[\ln\left|e^{x}-e^{-x}\right|\Big]_{\ln\frac{3}{2}}^{\ln 2}$$

$$I=\ln\left|e^{\ln 2}-e^{-\ln 2}\right|-\ln\left|e^{\ln\frac{3}{2}}-e^{-\ln\frac{3}{2}}\right|$$

$$=\ln\left|2-\frac{1}{2}\right|-\ln\left|\frac{3}{2}-\frac{2}{3}\right|$$

$$=\ln\frac{3}{2}-\ln\frac{5}{6}=\ln\frac{9}{6}-\ln\frac{5}{6}$$

$$=\ln\left|\frac{9}{6}\div\frac{5}{6}\right|$$

$$=\ln\frac{9}{5}$$

6)

$$z=\frac{(1+2i)^{2}}{7-i}$$

$$=\frac{(1+2i)^{2}(7+i)}{(7-i)(7+i)}$$

$$=\frac{(1+4i-4)(7+i)}{49+1}$$

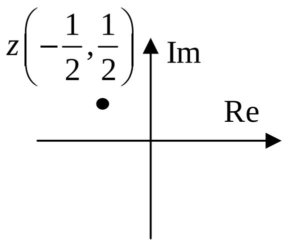
$$=\frac{7+28i-28+i-4-4i}{50}$$

$$=\frac{25i-25}{50}=-\frac{1}{2}+\frac{1}{2}i$$

$$|z|=\sqrt{2\left(\frac{1}{2}\right)^{2}}$$

$$=\sqrt{2\times\frac{1}{4}}$$

$$=\frac{1}{\sqrt{2}}$$



$$\arg z=\tan^{-1}\left[\frac{1}{2}\div\left(-\frac{1}{2}\right)\right]$$

$$=\tan^{-1}-1$$

$$=135^{o}=\frac{3\pi}{4}$$

7)

$$I=\int_{0}^{\sqrt{2}}\frac{x^{2}}{\sqrt{4-x^{2}}}dx$$

$$x=2\sin\theta$$

$$\frac{dx}{d\theta}=2\cos\theta$$

$$dx=2\cos\theta d\theta$$

$$x=\sqrt{2}\rightarrow\theta=\sin^{-1}\frac{1}{\sqrt{2}}=\frac{\pi}{4}$$

$$x=0\rightarrow\theta=\sin^{-1}0=0$$

$$I=\int_{0}^{\frac{\pi}{4}}\frac{(2\sin\theta)^{2}\,2\cos\theta}{\sqrt{4-(2\sin\theta)^{2}}}d\theta$$

$$=\int_{0}^{\frac{\pi}{4}}\frac{8\sin^{2}\theta\cos\theta}{\sqrt{4-4\sin^{2}\theta}}d\theta$$

$$=\int_{0}^{\frac{\pi}{4}}\frac{8\sin^{2}\theta\cos\theta}{2\sqrt{1-\sin^{2}\theta}}d\theta$$

$$=\int_{0}^{\frac{\pi}{4}}\frac{4\sin^{2}\theta\cos\theta}{\sqrt{\cos^{2}\theta}}d\theta$$

$$=-2\int_{0}^{\frac{\pi}{4}}-2\sin^{2}\theta d\theta$$

$$=-2\int_{0}^{\frac{\pi}{4}}-2\sin^{2}\theta d\theta$$

$$=-2\int_{0}^{\frac{\pi}{4}}\left(1-2\sin^{2}\theta-1\right)d\theta$$

$$=-2\int_{0}^{\frac{\pi}{4}}\left(\cos 2\theta-1\right)d\theta$$

$$=-2\left[\frac{1}{2}\sin 2\theta-\theta\right]_{0}^{\frac{\pi}{4}}$$

$$=-2\left(\frac{1}{2}\sin\frac{\pi}{2}-\frac{\pi}{4}\right)-0$$

$$=-2\left(\frac{1}{2}-\frac{\pi}{4}\right)$$

$$=\frac{\pi}{2}-1$$

8a)

$$(1+x)^5 = \sum_{r=0}^{5} \binom{5}{r} x^r$$
$$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

8b)

Let $x = -0.1$

$$0.9^5 = (1+x)^5 = (1-0.1)^5$$
$$= 1 + 5(-0.1) + 10(-0.1)^2 + 10(-0.1)^3 + 5(-0.1)^4 + (-0.1)^5$$
$$= 1 - 0.5 + 0.1 - 0.01 + 0.0005 - 0.00001$$
$$= 0.59049$$

9)

$$I = \int_0^1 x \tan^{-1} x^2 dx$$

$$u' = x \qquad u = \frac{1}{2}x^2$$
$$v = \tan^{-1} x^2 \qquad v' = \frac{2x}{1+x^4}$$
$$I = \left[uv - \int uv' dx \right]_0^1$$
$$= \left[\frac{1}{2}x^2 \tan^{-1} x^2 - \int \left(\frac{1}{2}x^2 \times \frac{2x}{1+x^4} \right) dx \right]_0^1$$
$$= \left[\frac{1}{2}x^2 \tan^{-1} x^2 - \int \frac{x^3}{1+x^4} dx \right]_0^1$$
$$= \left[\frac{1}{2}x^2 \tan^{-1} x^2 - \frac{1}{4}\int \frac{4x^3}{1+x^4} dx \right]_0^1$$
$$= \left[\frac{1}{2}x^2 \tan^{-1} x^2 - \frac{1}{4}\ln\left|1+x^4\right| \right]_0^1$$
$$= \frac{1}{2}\tan^{-1} 1 - \frac{1}{4}\ln|1+1| - 0$$
$$= \frac{1}{2} \times \frac{\pi}{4} - \frac{1}{4}\ln 2$$
$$= \frac{\pi}{8} - \frac{1}{4}\ln 2$$

10)

$$14654 = 11.1326 + 68$$
$$1326 = 19.68 + 34$$
$$68 = 2.34$$
$$\text{gcd} = 34$$

$$34 = 1326 - 19.68$$
$$= 1326 - 19(14654 - 11.1326)$$
$$= 1326 - 19.14654 + 209.1326$$
$$= 210.1326 - 19.14654$$

11)

$$y = x^{2x^2+1}$$
$$x = 1, y = 1$$

$$\ln y = \ln x^{2x^2+1} = (2x^2+1)\ln x$$
$$\frac{1}{y}\frac{dy}{dx} = 4x\ln x + \frac{2x^2+1}{x}$$
$$\frac{dy}{dx} = y\left(4x\ln x + \frac{2x^2+1}{x}\right)$$
$$= 4\ln 1 + 2 + 1$$
$$= 2 + 1$$
$$= 3$$

12)

$$r = p,\ a = p$$
$$S_n = \frac{a(1-r^n)}{1-r} = \frac{p(1-p^n)}{1-p}$$
$$S_{2n} = \frac{p(1-p^{2n})}{1-p}$$
$$S_{2n} = 65S_n$$
$$\frac{p(1-p^{2n})}{1-p} = 65 \times \frac{p(1-p^n)}{1-p}$$
$$1 - p^{2n} = 65(1-p^n)$$
$$= 65 - 65p^n$$
$$p^{2n} - 65p^n + 64 = 0$$
$$(p^n - 64)(p^n - 1) = 0$$
$$p^n = 64$$

$p^n = 1$ is discarded since $n = 0$...

... impossible as $S_k = \sum_{j=1}^{k} a_k$ implies minimum $n = 1$

also $p \neq 1$ as then S_n is undefined

$$a_3 = 2p = p^3$$
$$2 = p^2$$
$$p = \sqrt{2}$$
$$p^n = \left(\sqrt{2}\right)^n = 64$$
$$2^6 = 64 \ \therefore n = 12$$

13)

$$f(x)=\frac{x^2+2x}{x^2-1}$$

VA: $x=1,\ x=-1$

$$\begin{array}{r|l} & \quad\quad 1 \\ x^2-1 & x^2+2x \\ & \underline{x^2 \quad\quad -1} \\ & \quad 2x+1 \end{array}$$

$$f(x)=1+\frac{1+2x}{x^2-1}$$

$\therefore$ NVA: $y=1$

$$f'(x)=\frac{2(x^2-1)-2x(1+2x)}{(x^2-1)^2}$$

$$=\frac{2x^2-2-2x-4x^2}{(x^2-1)^2}$$

$$=\frac{-2x^2-2-2x}{(x^2-1)^2}$$

$$=-2\times\frac{x^2+x+1}{(x^2-1)^2}$$

$x^2+1>x$ so x^2+x+1 is always positive, $x\in\mathbb{R}$

$(x^2-1)^2$ is always positive

so $f'(x)$ is always negative

and there is no solutions for $f'(x)=0$

$\Leftrightarrow$ f (x) is always decreasing

13i)

$$f(x)=\frac{x^2+2x}{x^2-1}=0$$

$$x^2+2x=0$$

$$x(x+2)=0$$

$$x=-2,0$$

$$(-2,0),(0,0)$$

13ii)

$$f(x)=\frac{x^2+2x}{x^2-1}=1$$

$$x^2+2x=x^2-1$$

$$2x=-1$$

$$x=-\frac{1}{2}$$

$$(-0.5,1)$$

See appendix for graph

14)

$$\frac{x^2+6x-4}{(x+2)^2(x-4)}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{x-4}$$

$$x^2+6x-4=A(x+2)(x-4)+B(x-4)+C(x+2)^2$$

$x=-2$:

$$4-12-4=-6B=-12$$

$$B=2$$

$x=4$:

$$16+24-4=36C=36$$

$$C=1$$

$$x^2=Ax^2+Cx^2$$

$$1=A+1$$

$$A=0$$

$$\frac{x^2+6x-4}{(x+2)^2(x-4)}=\frac{2}{(x+2)^2}+\frac{1}{x-4}$$

$x=0$:

$$f(x)=\frac{1}{x-4}=-\frac{1}{4}$$

$$f'(x)=\frac{-1}{(x-4)^2}=-\frac{1}{16}$$

$$f''(x)=\frac{2}{(x-4)^3}=\frac{2}{-64}=-\frac{1}{32}$$

$$f(x)=-\frac{1}{4}-\frac{x}{16}-\frac{x^2}{64}\ldots$$

$$g(x)=\frac{2}{(x+2)^2}=\frac{2}{4}=\frac{1}{2}$$

$$g'(x)=\frac{-4}{(x+2)^3}=\frac{-4}{8}=-\frac{1}{2}$$

$$g''(x)=\frac{12}{(x+2)^4}=\frac{12}{16}=\frac{3}{4}$$

$$g(x)=\frac{1}{2}-\frac{x}{2}+\frac{3x^2}{8}+\ldots$$

$$\frac{x^2+6x-4}{(x+2)^2(x-4)}=f(x)+g(x)$$

$$=-\frac{1}{4}-\frac{x}{16}-\frac{x^2}{64}+\frac{1}{2}-\frac{x}{2}+\frac{3x^2}{8}+\ldots$$

$$=\frac{1}{4}-\frac{9x}{16}+\frac{23x^2}{64}+\ldots$$

15a)

$$(x+1)\frac{dy}{dx}-3y=(x+1)^4$$

$$\frac{dy}{dx}-\frac{3y}{x+1}=(x+1)^3$$

$$I(x)=e^{-3\int\frac{dx}{x+1}}=e^{-3\ln(x+1)}=\frac{1}{(x+1)^3}$$

$$I(x)\left(\frac{dy}{dx}-\frac{3y}{x+1}\right)=1=\frac{d(I(x)y)}{dx}$$

$$I(x)y=\int 1\,dx$$

$$=x+C$$

$$y=(x+1)^3x+(x+1)^3C$$

$$16=2^3+2^3C$$

$$8=8C$$

$$C=1$$

$$y=(x+1)^3x+(x+1)^3$$

$$=(x+1)^3(x+1)$$

$$=(x+1)^4$$

15b)

We note that $y(-x)=(1-x)^4$

and also that for $(x+1)^4=(1-x)^4$

$x+1=\pm(1-x)\rightarrow$ only solution $x=0$

This shows $(1-x)^4$ is the reflection of $(x+1)^4$ on the y-axis

$$(x+1)^4=0$$

$$x=-1$$

Thus $I=2\int_{-1}^{0}(x+1)^4\,dx$

$$=\frac{2}{5}\left[(x+1)^5\right]_{-1}^{0}$$

$$=\frac{2}{5}-0$$

$$=\frac{2}{5}$$

16a)

$$\left(\begin{array}{ccc|c}1&1&-1&6\\2&-3&2&2\\-5&2&-4&1\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}1&1&-1&6\\0&-5&4&-10\\-5&2&-4&1\end{array}\right)$$

$$\rightarrow\left(\begin{array}{ccc|c}1&1&-1&6\\0&-5&4&-10\\0&7&-9&31\end{array}\right)$$

$$\rightarrow\left(\begin{array}{ccc|c}1&1&-1&6\\0&-35&28&-70\\0&35&-45&155\end{array}\right)$$

$$\rightarrow\left(\begin{array}{ccc|c}1&1&-1&6\\0&-35&28&-70\\0&0&-17&85\end{array}\right)$$

$$\rightarrow\left(\begin{array}{ccc|c}1&1&-1&6\\0&-5&4&-10\\0&0&-1&5\end{array}\right)$$

$$-z=5$$

$$z=-5$$

$$-5y+4z=-10$$

$$-5y-20=-10$$

$$-5y=10$$

$$y=-2$$

$$x+y-z=6$$

$$x-2+5=6$$

$$x=3$$

16b)

Let $x=\lambda$

$$\lambda+y-z=6$$

$$y=6+z-\lambda$$

$$3y=18+3z-3\lambda$$

$$2\lambda-3y+2z=2$$

$$3y=2\lambda+2z-2=18+3z-3\lambda$$

$$z=5\lambda-20$$

$$y=6+z-\lambda$$

$$=6+5\lambda-20-\lambda$$

$$=4\lambda-14$$

16c)

$$d_L = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$$

$$|d_L| = \sqrt{1+4^2+5^2}$$
$$= \sqrt{42}$$

$$n = \begin{pmatrix} -5 \\ 2 \\ -4 \end{pmatrix}$$

$$|n| = \sqrt{2^2+4^2+5^2}$$
$$= \sqrt{45}$$

$$\sin\theta = \frac{d_L.n}{|d_L||n|}$$
$$= \frac{-5+8-20}{\sqrt{42}\sqrt{45}}$$
$$= -\frac{17}{\sqrt{1890}}$$
$$= -0.391$$
$$\theta = -23.02^o \ldots \to \text{acute angle} = 23.02^o$$

Appendix – 13) Graph